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Absorption probabilities for a random walk between two partially absorbing boundaries: I

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Abstract. A biased random walk on a linear lattice bounded by one or two partially absorbing boundaries is considered. At the boundaries, the particle is either lost from the system or turned back, and reduces to the classical problem of a random walk with absorbing and/or reflecting barriers. The probabilities of ultimate absorption at the boundaries are obtained, as well as the conditional mean for the number of steps before stopping given the absorption at a specified barrier. The symmetries for the conditional mean are also given. Previous formulae are found to be in error.

1. Introduction

Consider a biased random walk on the finite set $S = \{0, 1, \ldots, L\}$, in the presence of partially absorbing boundaries at 0 and L, such that a particle when away from the boundaries, say at $j \in S \setminus \{0, L\}$, moves regularly to adjacent integers, with probability $P(j)$, $0 < P(j) < 1$, to the higher integer and probability $1 - p(j)$ to the lower integer, when the particle hits the boundaries it is reflected or absorbed with respective probabilities ρ and $1 - \rho$ at the origin and ω and $1 - \omega$ at L. Physically, this corresponds to the situation when, at the boundary 0 (boundary L) the particle is either lost from the system with probability $1-\rho$ (probability $1-\omega$) or turned back with probability ρ (probability ω), and reduces to the classical problem of a random walk in the presence of two perfectly absorbing barriers for $\rho = \omega = 0$, two reflecting barriers for $0 < \rho \leqslant 1$, $0 < \omega \leqslant 1$, two different barriers one of which is perfectly absorbing and the other is a reflector for $\rho = 0$, $\omega \leqslant 1$ (or $\rho \leqslant 1$, $\omega = 0$). Random walk problems on regular lattices have a long and notable history. They serve as abstract representations of many random processes.

We seek explicit expressions for the probabilities of ultimate absorption at the boundaries, as well as the conditional mean for the number of steps before stopping given the absorption at a specified boundary. The symmetries for the conditional mean are obtained. The generating function of the n-step probability that the particle is at a specified location after n steps, when $\rho = \omega$, has been computed by Percus (1985); however, her calculations contained some errors. The correct formulae can be found in section 3.

The outline of this paper is as follows. In section 2, we derive general formulae for the probabilities of ultimate absorption at $k \in \{0, L\}$. In section 3, we present explicit expressions for the mean number of steps taken before absorption, as well as the conditioned mean duration. Symmetries for the conditioned mean of time to absorption are found in section 4.

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2. Absorption probabilities

Let $q(k|i)$ be the probability of ultimate absorption at $k \in \{0, L\}$, given that i was its initial position. Then it can be easily shown by conditioning with respect to the first move that the absorption probabilities $q(0|i)$ satisfy

$$q(0|i) = p(i)q(0|i+1) + (1-p(i))q(0|i-1) \qquad i \in S \setminus \{0, L\} \tag{2.1}$$

$$q(0|0) = (1-\rho) + \rho q(0|1) \tag{2.2}$$

$$q(0|L) = \omega q(0|L-1). \tag{2.3}$$

Solving systematically, we obtain

$$q(0|i) = 1 - \frac{(1-\omega)\left[\rho/(1-\rho) + \sum_{g=1}^{i}\prod_{k=1}^{g-1} x_k\right]}{(1-\omega)\rho/(1-\rho) + \sum_{g=1}^{L}\prod_{k=1}^{g-1} x_k - \omega\sum_{g=1}^{L-1}\prod_{k=1}^{g-1} x_k} \qquad i \in S \tag{2.4}$$

(by convention, $\prod_{k=1}^{0} x_k = 1$).

The probability, $q(L|i)$ may be obtained from (2.4) by interchanging $p(k)$ and $1 - p(k)$, ρ and ω, and replacing i by $L - i$. Thus

$$q(L|i) = \frac{(1-\omega)\left[\rho/(1-\rho) + \sum_{g=1}^{i}\prod_{k=1}^{g-1} x_k\right]}{(1-\omega)\rho/(1-\rho) + \sum_{g=1}^{L}\prod_{k=1}^{g-1} x_k - \omega\sum_{g=1}^{L-1}\prod_{k=1}^{g-1} x_k} \qquad i \in S. \tag{2.5}$$

Clearly, $q(0|i) + q(L|i) = 1$.

If $\rho = \omega = 0$, then we obtain the well known formulae (Parzen 1962, p 233) that

$$q(0|i) = \left(\sum_{g=i}^{L-1}\prod_{k=1}^{g} x_K\right)\bigg/\left(\sum_{g=0}^{L-1}\prod_{k=1}^{g} x_K\right) \tag{2.6}$$

$$q(L|i) = \left(\sum_{g=0}^{i-1}\prod_{k=1}^{g} x_K\right)\bigg/\left(\sum_{g=0}^{L-1}\prod_{k=1}^{g} x_K\right) \tag{2.7}$$

where $x_k = (1/p(k) - 1)$.

When spatial homogeneity is present, on setting $p(k) = p, k \in S \setminus \{0, L\}$. Formulae (2.4) and (2.5) become, for $i \in S$

$$q(0|i) = \begin{cases} \left[\left(\dfrac{1-p}{p}\right)^{i} - \dfrac{((1-p)/p) - \omega}{1-\omega}\left(\dfrac{1-p}{p}\right)^{L-1}\right] & \\ \qquad \times\left[\dfrac{1-\rho((1-p)/p)}{1-\rho}\right. & \\ \qquad \left. - \dfrac{((1-p)/p) - \omega}{1-\omega}\left(\dfrac{1-p}{p}\right)^{L-1}\right]^{-1} & \text{if} \quad p \neq \frac{1}{2} \\ \dfrac{L - i + \omega/(1-\omega)}{L + \rho/(1-\rho) + \omega/(1-\omega)} & \text{if} \quad p = \frac{1}{2} \end{cases} \tag{2.8}$$

and

$$q(L|i) = \begin{cases} \left[\dfrac{1-\rho((1-p)/p)}{1-\rho} - \left(\dfrac{1-p}{p}\right)^{i}\right]\left[\dfrac{1-\rho((1-p)/p)}{1-\rho}\right. & \\ \qquad \left. - \left(\dfrac{(1-p)/p - \omega}{1-\omega}\right)\left(\dfrac{1-p}{p}\right)^{L-1}\right]^{-1} & \text{if} \quad p \neq \frac{1}{2} \\ \dfrac{i + \rho/(1-\rho)}{L + \rho/(1-\rho) + \omega/(1-\omega)} & \text{if} \quad p = \frac{1}{2}. \end{cases} \tag{2.9}$$

We see that with the appropriate change of notation, expressions (2.8) and (2.9) in the special case $\rho = \omega = 0$ agree with those of Feller (1968, pp 344–9; see also Cox and Miller 1965, Beyer and Waterman 1977, El-Shehawey 1994), and agree with that of Percus (1985) (formulae (3.1) and (3.2), respectively, therein) in the case $\rho = \omega$.

The one-boundary case with the assumption that there is one partially absorbing barrier at 0 and the other at ∞, can be obtained from (2.8) and (2.9) by taking the limit as $L \to \infty$:

$$q(0|i) = \begin{cases} \dfrac{1-\rho}{1-\rho((1-p)/p)}\left(\dfrac{1-p}{p}\right)^i & \text{if} \quad p > \frac{1}{2} \\ 1 & \text{if} \quad p \leqslant \frac{1}{2} \end{cases} \tag{2.10}$$

and

$$q(L|i) = \begin{cases} 1 - \dfrac{1-\rho}{1-\rho((1-p)/p)}\left(\dfrac{1-p}{p}\right)^i & \text{if} \quad p > \frac{1}{2} \\ 0 & \text{if} \quad p \leqslant \frac{1}{2}. \end{cases} \tag{2.11}$$

Formulae (2.10) and (2.11) in the special case $\rho = 1$ for $p \leqslant \frac{1}{2}$, agree with that of Kac (1954, p 295); see also Percus (1985).

3. Conditioned mean duration

Suppose the duration of the walk has a finite mean m_i given the starting point $i \in S$. The mean absorption times $\{m_i, i \in S\}$ satisfies

$$p(j)m_{j+1} - m_j + (1-p(j))m_{j-1} = -1 \qquad j \in S \setminus \{0, L\} \tag{3.1}$$

$$\rho m_1 - m_0 = -\rho \tag{3.2}$$

$$\omega m_{L-1} - m_L = -\omega. \tag{3.3}$$

Solving recursively, one obtains

$$\begin{aligned} M_{j+1} = {} & \prod_{k=0}^{m}\left(\frac{1-p(j-k)}{p(j-k)}\right)M_{j-m} \\ & -\frac{1}{p(j)}\left[1 + \frac{1-p(j)}{p(j-1)} + \cdots + \prod_{k=1}^{m}\left(\frac{1-p(j-k+1)}{p(j-k)}\right)\right] \end{aligned} \tag{3.4}$$

with

$$M_j = m_j - m_{j-1} \qquad j \in S \setminus \{0\}. \tag{3.5}$$

From (3.4) and (3.5) one could obtain an expression for m_i.

Assume spatial homogeneity is present, on setting $p(k) = p$, $k \in S\backslash\{0, L\}$. The following equation could be achieved:

$$m_i = m_0 + \sum_{j=1}^{i} M_j = \begin{cases} \left[\dfrac{1-((1-p)/p)^i - [\rho(1-p)/p][1-((1-p)/p)^{i-1}]}{\rho(2-1/p)}\right]m_0 & \\ +\dfrac{[2(1-p)/p][1-((1-p)/p)^i]}{(2-1/p)^2} - \dfrac{i}{2p-1} & \text{if} \quad p \neq \frac{1}{2} \\ \left[1 + \left(\dfrac{1-\rho}{\rho}\right)i\right]m_0 - i^2 & \text{if} \quad p = \frac{1}{2}. \end{cases} \tag{3.6}$$

From (3.3) and (3.6), we have

$$m_0 = \begin{cases} \rho\Big[\Big(\frac{1-\omega}{p}\Big)L - \frac{2((1-p)/p-\omega)}{2-1/p}\Big[1-\Big(\frac{1-p}{p}\Big)^L\Big]\Big] \\ \quad \times\Big[(1-\omega)\Big(1-\frac{\rho(1-p)}{p}\Big) \\ \quad -(1-\rho)\Big(\frac{1-p}{p}-\omega\Big)\Big(\frac{1-p}{p}\Big)^{L-1}\Big]^{-1} & \text{if} \quad p \neq \frac{1}{2} \\ \frac{[\rho L/(1-\rho)][L+2\omega/(1-\omega)]}{L+\rho/(1-\rho)+\omega/(1-\omega)} & \text{if} \quad p = \frac{1}{2}. \end{cases} \tag{3.7}$$

Therefore, the mean number of steps taken before absorption, given the starting point $i \in S$, is

$$m_i = \begin{cases} \frac{1}{2p-1}\Big\{\frac{(L-x)[1-((1-p)/p)^i - [\rho(1-p)/p][1-((1-p)/p)^{i-1}]]}{1-[\rho(1-p)/p]-[(1-\rho)((1-p)/p-\omega)/(1-\omega)]((1-p)/p)^{L-1}} \\ \quad -i-\frac{2\rho(1-p)}{1-\rho}\Big\} & \text{if} \quad p \neq \frac{1}{2} \\ \frac{L(L+2\omega/(1-\omega))(i+\rho/(1-\rho))}{L+\rho/(1-\rho)+\omega/(1-\omega)} - i^2 & \text{if} \quad p = \frac{1}{2} \end{cases} \tag{3.8}$$

where

$$x = \frac{2p}{(1-\omega)(2-1/p)}\Big[\frac{\rho(1-\omega)}{1-\rho}\Big(\frac{1-p}{p}\Big)^2 + \Big(\frac{\omega-\rho}{1-\rho}\Big)\Big(\frac{1-p}{p}\Big) - \omega\Big].$$

It is interesting to note that expression (3.8) agrees well with the results reported by Percus (1985) in the case $p \neq \frac{1}{2}$. On the other hand, at $p = \frac{1}{2}$ the expression is not well fitted and should be revised to be

$$\frac{Ls(L-s)(1-\rho)}{L(1-\rho)+2\rho} + \frac{\rho(L-1)}{1-\rho}$$

instead of

$$s(L-s) + \frac{\rho L}{1-\rho}.$$

Formula (3.8) also agrees with the well known result for a random walk with absorbing barriers ($\rho = \omega = 0$) (see Parzen 1962, p 240).

In the one-boundary case with one partially absorbing barrier at 0 and the other at ∞, the mean duration of the walk can be obtained from (3.8) by taking the limit as $L \to \infty$:

$$m_i = \begin{cases} \Big[i + \frac{2\rho(1-p)}{1-\rho}\Big]\Big(\frac{1}{1-2p}\Big) & \text{if} \quad p < \frac{1}{2} \\ \infty & \text{if} \quad p \geqslant \frac{1}{2}. \end{cases} \tag{3.9}$$

Let $E_i^{(k)}$ be the conditional mean for the number of steps taken until being absorbed given absorption occurs at k, $k \in \{0, L\}$, then the mean duration is equal to the weighted sum of the conditional means:

$$m_i = q(L|i)E_i^{(L)} + q(0|i)E_i^{(0)}. \tag{3.10}$$

If we call $P_n(k|i)$, the probability that the particle is at the point k, $k \in \{0, L\}$, after n steps, given the initial position i, then

$$q(0|i) = (1-\rho)\sum_{n=0}^{\infty} p_n(0|i) \qquad \text{and} \qquad q(L|i) = (1-\omega)\sum_{n=0}^{\infty} p_n(L|i). \tag{3.11}$$

The product $q(k|i)E_i^{(L)}$, which we rename $m_i^{(k)}$, satisfies

$$\begin{aligned} m_i^{(k)} &= q(k|i)E_i^{(k)} = \sum_{n=0}^{\infty} n p_n(k|i) \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\varphi_k(z|i)|_{z=1} \qquad k = 0, L \end{aligned} \tag{3.12}$$

where

$$\varphi_k(z|i) = \sum_{n=0}^{\infty} z^n p_n(k|i) \qquad |z| < 1.$$

In the case $p = \frac{1}{2}$, $m_i^{(0)}$ satisfies the following recurrence relation:

$$\tfrac{1}{2}m_{i+1}^{(0)} - m_i^{(0)} + \tfrac{1}{2}m_{i-1}^{(0)} = -q(0|i) \qquad i \in S \setminus \{0, L\} \tag{3.13}$$

with the boundary conditions

$$\rho m_1^{(0)} - m_0^{(0)} = -\rho \tag{3.14}$$

$$\omega m_{L-1}^{(0)} - m_L^{(0)} = -\omega \tag{3.15}$$

where $q(0|i)$ is given by formula (2.8).

The unique solution of (3.13) with (3.14) and (3.15) is given by

$$\begin{aligned} m_i^{(0)} = &\left\{ i\left[i^2 - 3\left(L + \frac{\omega}{1-\omega}\right)i - \frac{1+2\rho}{1-\rho}\right] + \frac{(i+\rho/(1-\rho))\psi_L(\rho,\omega)}{L+\rho/(1-\rho)+\omega/(1-\omega)}\right\} \\ &\times\left[3\left(L + \frac{\rho}{1-\rho} + \frac{\omega}{1-\omega}\right)\right]^{-1} \qquad i \in S \end{aligned} \tag{3.16}$$

and hence

$$\begin{aligned} E_i^{(0)} = &\left\{ i\left[i^2 - 3\left(L + \frac{\omega}{1-\omega}\right)i - \frac{1+2\rho}{1-\rho}\right] + \frac{(i+\rho/(1-\rho))\psi_L(\rho,\omega)}{L+\rho/(1-\rho)+\omega/(1-\omega)}\right\} \\ &\times\left[3\left(L - i + \frac{\omega}{1-\omega}\right)\right]^{-1} \qquad i \in S \end{aligned} \tag{3.17}$$

where

$$\psi_L(\rho,\omega) = \left(L + \frac{2\omega}{1-\omega}\right)\left[2L^2 + \left(\frac{1+2\omega}{1-\omega}\right)L + \frac{3\rho}{1-\rho}\right] - \frac{L(L-1)}{1-\omega}. \tag{3.18}$$

For $E_i^{(L)}$ in the case $p = \frac{1}{2}$, we note that $m_i^{(L)} = m_{L-i}^{(0)}$ with interchanging ρ and ω. Thus

$$\begin{aligned} m_i^{(L)} = &\left\{ (L-i)\left[(i-L)\left(i + 2L + \frac{3\rho}{1-\rho}\right) - \frac{1+2\omega}{1-\omega}\right] + \frac{(L-i+\omega/(1-\omega))\,\psi_L(\omega,\rho)}{L+\rho/(1-\rho)+\omega/(1-\omega)}\right\} \\ &\times\left[3\left(L + \frac{\rho}{1-\rho} + \frac{\omega}{1-\omega}\right)\right]^{-1} \qquad i \in S \end{aligned} \tag{3.19}$$

and

$$E_i^{(L)} = \left\{(L-i)\left[(i-L)\left(i+2L+\frac{3\rho}{1-\rho}\right)-\frac{1+2\omega}{1-\omega}\right]+\frac{(L-i+\omega/(1-\omega))\,\psi_L(\omega,\rho)}{L+\rho/(1-\rho)+\omega/(1-\omega)}\right\}$$
$$\times\left[3\left(i+\frac{\rho}{1-\rho}\right)\right]^{-1} \qquad i \in S \tag{3.20}$$

where

$$\psi_L(\omega,\rho) = \left(L+\frac{2\rho}{1-\rho}\right)\left[2L^2+\left(\frac{1+2\rho}{1-\rho}\right)L+\frac{3\omega}{1-\omega}\right]-\frac{L(L-1)}{1-\rho}. \tag{3.21}$$

In the case $p \neq \frac{1}{2}$, $m_i^{(0)}$ obeys the following recurrence relation:

$$pm_{i+1}^{(0)} - m_i^{(0)} + (1-p)m_{i-1}^{(0)} = -q(0|i) \qquad i \in S\setminus\{0, L\} \tag{3.22}$$

with the boundary conditions (3.14) and (3.15), where $q(0|i)$ is given by (2.8). Rather than proceeding as above, we use an alternate method of finding $m_i^{(0)}$.

Following Feller (1968) we deduce that for the generating functions of the n-step probability, $p_n(k|i)$, the particle is at location k after n steps, given the starting point i, for $k = 0$, and $k = L$ are given by

$$\varphi_0(z|i) = \frac{((1-p)/p)^i}{D}\left[\lambda_1^{L-i}-\lambda_2^{L-i}-\omega z\left(\lambda_1^{L-i-1}-\lambda_2^{L-i-1}\right)\right] \tag{3.23}$$

and

$$\varphi_L(z|i) = \frac{1}{D}\left[\lambda_1^{i}-\lambda_2^{i}-\frac{\rho z(1-p)}{p}\left(\lambda_1^{i-1}-\lambda_2^{i-1}\right)\right] \tag{3.24}$$

where

$$D = \lambda_1^L-\lambda_2^L - z\left(\omega+\frac{\rho(1-p)}{p}\right)\left(\lambda_1^{L-1}-\lambda_2^{L-1}\right)+\frac{\rho\omega z^2(1-p)}{p}\left(\lambda_1^{L-2}-\lambda_2^{L-2}\right) \tag{3.25}$$

and

$$\begin{aligned}\lambda_1 = \lambda_1(z) &= \frac{1}{2zp}\left(1+\sqrt{1-4p(1-p)z^2}\right)\\ \lambda_2 = \lambda_2(z) &= \frac{1}{2zp}\left(1-\sqrt{1-4p(1-p)z^2}\right).\end{aligned} \tag{3.26}$$

If we rewrite formulae (3.23)–(3.26) in the case $\rho = \omega$ and replacing λ_j by $\sqrt{q/p}\,\theta_j$ $(j = 1, 2, q = 1-p)$, z by θ and i by s, we obtain

$$\begin{aligned}\varphi_0(\lambda|s) &= \sum_{n=0}^{\infty}\lambda^n p_n(0|s)\\ &= \frac{\left(\sqrt{q/p}\right)^s}{\Delta}\left\{\lambda\rho\sqrt{\frac{p}{q}}\left(\theta_1^{L-s-1}-\theta_2^{L-s-1}\right)-\left(\theta_1^{L-s}-\theta_2^{L-s}\right)\right\}\end{aligned} \tag{3.27}$$

$$\begin{aligned}\varphi_L(\lambda|s) &= \sum_{n=0}^{\infty}\lambda^n p_n(L|s)\\ &= \frac{\left(\sqrt{p/q}\right)^{L-s}}{\Delta}\left\{\lambda\rho\sqrt{\frac{q}{p}}\left(\theta_1^{s-1}-\theta_2^{s-1}\right)-\left(\theta_1^{s}-\theta_2^{s}\right)\right\}\end{aligned} \tag{3.28}$$

where

$$\Delta = -\left[(\lambda\rho)^2\left(\theta_1^{L-2} - \theta_2^{L-2}\right) - \frac{\lambda\rho}{\sqrt{pq}}\left(\theta_1^{L-1} - \theta_1^{L-2}\right) + \theta_1^L - \theta_2^L\right]. \qquad (3.29)$$

We see that formula (3.28) agrees with formula (2.23) in Percus (1985); however, formulae (3.27) and (3.29) differ from Percus' (1985) formulae (2.22) and (2.17), the misprints being:

- in (2.17), Q_2^{L-1} should be θ_2^{L-1},
- in (2.22), $(p/q)^{S/2}$ should be replaced by $(q/p)^{S/2}$,

and $\sum_{n=0}^{\infty} p_n(0)\lambda^n$ on p 598 should be

$$\frac{(q/p)^{S/2}\left[\lambda\rho\sqrt{p/q}\left(\theta_2^{S-1} - \theta_2^{2L-S-1}\right) - \left(\theta_2^S - \theta_2^{2L-s}\right)\right]}{\left(1-\lambda^2\rho^2\right)\left(\theta_2^2 - \theta_2^{2L-2}\right) - \left(1-\lambda^2\rho\right)\left(1+\theta_2^2\right)\left(1-\theta_2^{2L-2}\right)}.$$

Differentiating $\varphi_0(z|i)$ with respect to z, setting $z = 1$ and noting for $p > \frac{1}{2}$,

$$\begin{aligned} &\lambda_1(1) = 1 \qquad \lambda_2(1) = \frac{1-p}{p} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\lambda_j(z) = \frac{1-p\left(1-\lambda_j^2\right)}{1-2pz\lambda_j} \qquad j = 1, 2 \\ &\left.\frac{\mathrm{d}}{\mathrm{d}z}\lambda_1(z)\right|_{z=1} = \frac{1}{1-2p} \qquad \left.\frac{\mathrm{d}}{\mathrm{d}z}\lambda_2(z)\right|_{z=1} = \frac{1-p}{p(2p-1)} \end{aligned} \qquad (3.30)$$

and for $p < \frac{1}{2}$, the same results hold with subscripts 1 and 2 interchanged, we obtain

$$m_i^{(0)} = \frac{1}{(2p-1)\alpha}\left[(L-i)\left(a^i + \frac{a-\omega}{1-\omega}a^{L-1}\right) + \frac{2\omega p}{1-\omega}(a^i + a^L) + \frac{\beta}{\alpha}\left(a^i - \frac{a-\omega}{1-\omega}a^{L-1}\right)\right] \qquad (3.31)$$

where α and β are, functions of L, ρ and ω, given by

$$\begin{aligned} &\alpha = \alpha(L, \rho, \omega) = \frac{a-\omega}{1-\omega}a^{L-1} - \frac{1-\rho a}{1-\rho} \qquad a = \frac{1-p}{p} \\ &\beta = \beta(L, \rho, \omega) = L\left(\frac{a-\omega}{1-\omega}a^{L-1} + \frac{1-\rho a}{1-\rho}\right) \\ &\qquad + \frac{2p}{(1-\rho)(1-\omega)}\left[(\omega + \rho a)(1 + a^L) - 2\rho\omega a(1 + a^{L-1})\right]. \end{aligned} \qquad (3.32)$$

Thus from (2.8), (3.12) and (3.31) we can deduce

$$E_i^{(0)} = \frac{1}{1-2p}\left[\frac{(L-i)\left(a^i + [(a-\omega)/(1-\omega)]a^{L-1}\right) + [2\omega p/(1-\omega)](a^i + a^L)}{a^i - [(a-\omega)/(1-\omega)]a^{L-1}} + \frac{\beta}{\alpha}\right]. \qquad (3.33)$$

With regard to $E_i^{(L)}$, recall that

$$m_i^{(L)} = \frac{1}{\alpha}\left(a^i - \frac{1-\rho a}{1-\rho}\right)E_i^{(L)}. \qquad (3.34)$$

The mean $m_i^{(L)}$ is given, however, by $m_{L-i}^{(0)}$ if in addition the roles of p and $1-p$ (or a and a^{-1}), ρ and ω are interchanged in formula (3.31):

$$m_i^{(L)} = \frac{1}{(2p-1)\alpha}\left[i\left(a^i + \frac{1-\rho a}{1-\rho}\right) + \frac{2\rho(1-p)}{1-\rho}\left(a^i + 1\right) - \frac{\beta}{\alpha}\left(a^i - \frac{1-\rho a}{1-\rho}\right)\right]. \qquad (3.35)$$

Using (3.34) and (3.33) we obtain

$$E_i^{(L)} = \frac{1}{2p-1}\left[\frac{i\left(a^i + (1-\rho a)/(1-\rho)\right) + [2\rho(1-p)/(1-\rho)](a^i+1)}{a^i - (1-\rho a)/(1-\rho)} - \beta/\alpha\right]. \quad (3.36)$$

Formulae (3.17), (3.20), (3.32) and (3.35) agree with the well known results for a random walk between two perfectly absorbing barriers ($\rho = \omega = 0$) (see Stern 1975):

$$E_i^{(0)} = \begin{cases} \dfrac{i}{3}(2L-i) & \text{if} \quad p = \frac{1}{2} \\ \dfrac{(2p-1)^{-1}}{a^i - a^L}\left[i(a^i + a^L) + 2L\left(\dfrac{a^{L+i} - a^L}{1-a^L}\right)\right] & \text{if} \quad a = \dfrac{1-p}{p} \neq 1 \end{cases} \quad (3.37)$$

and

$$E_i^{(L)} = \begin{cases} \frac{1}{3}(L^2 - i^2) & \text{if} \quad p = \frac{1}{2} \\ \dfrac{(2p-1)^{-1}}{1-a^i}\left[(L-i)(a^i+1) + 2L\left(\dfrac{a^i - a^L}{a^L - 1}\right)\right] & \text{if} \quad a = \dfrac{1-p}{p} \neq 1. \end{cases} \quad (3.38)$$

4. Symmetries for the conditioned mean of time to absorption

When symmetry for partially absorbing barriers ($\rho = \omega$) is present, conditional on the absorption on L, the conditional mean for the number of steps to absorption from $i \in S$ is the same as the conditional mean of the number of steps to absorption from $L - i$ conditional on absorption in 0, with interchanging p and $1 - p$. Hence if $L = 2i$, in other words when the random walks starts at a point equidistant from 0 and L, the conditional mean $E_i^{(k)}$, $k = 0, L$ in the symmetrical case $p = \frac{1}{2}$, is independent of k, which is

$$E_i^{(k)} = i^2 + \frac{\rho}{1-\rho}[2i+1] \qquad k \in \{0, L\}. \quad (4.1)$$

Formula (4.1) follows from (3.17), (3.20) and (3.21) using the substitutions $L = 2i$ and $\rho = \omega$.

In the case $p \neq \frac{1}{2}$,

$$E_i^{(k)} = \frac{1}{2p-1}\left[\frac{i\left(a^i + (1-\rho a)/(1-\rho)\right) + [2\rho(1-p)/(1-\rho)](a^i+1)}{a^i - (1-\rho a)/(1-\rho)} - \frac{\beta(2i,\rho,\rho)}{\alpha(2i,\rho,\rho)}\right]$$
$$k \in \{0, L\}. \quad (4.2)$$

Expression (4.2) is derived by setting $L = 2i$, $\rho = \omega$ and interchanging p and $1 - p$ 'but p is still the probability of a right step' (replace a by a^{-1}) in (3.33), (3.36) and (3.32).

Formulae (4.1) and (4.2) agree with the results of a random walk with two perfectly absorbing boundaries ($\rho = \omega = 0$, $L = 2i$) (see, for example, Stern 1975):

$$E_i^{(k)} = \begin{cases} i^2 & \text{if} \quad p = \frac{1}{2} \quad k \in \{0, L\} \\ \dfrac{i}{2p-1}\left[\dfrac{1 - ((1-p)/p)^i}{1 + ((1-p)/p)^i}\right] & \text{if} \quad p \neq \frac{1}{2} \quad k \in \{0, L\}. \end{cases} \quad (4.3)$$

5. Conclusion

We have obtained exact analytical expressions for absorption probabilities and the mean time before absorption for a nearest-neighbour random walk on a finite set of integers with partially absorbing barriers. The behaviour for semi-infinite intervals is investigated. The paper also corrected some erroneous formulae of Percus for the spatially homogeneous equations. Symmetries for conditioned mean time to absorption are also given.

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References

Beyer W A and Waterman M S 1977 Symmetries for conditioned ruin problems *Math. Mag.* **50** 42–5

Cox D R and Miller H D 1965 *The Theory of Stochastic Processes* (London: Methuen)

El-Shehawey M A 1994 On the frequency count for a random walk with absorbing boundaries: a carcinogenesis example. I *J. Phys. A: Math. Gen.* **27** 7035–46

Feller W 1968 *An Introduction to Probability Theory and its Applications* vol 1, 3rd edn (New York: Wiley)

Kac M 1954 Random walk and the theory of Brownian motion *Selected Papers on Noise and Stochastic Processes* ed N Wax (New York: Dover)

Parzen E 1962 *Stochastic Process* (London: Holden-Day)

Percus O E 1985 Phase transition in one-dimensional random walk with partially reflecting boundaries *Adv. Appl. Prob.* **17** 594–606

Stern F 1975 Conditional expectation of the duration in the classical ruin problem *Math. Mag.* **48** 200–3